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RIEMANNIAN 4-SYMMETRIC SPACES

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ABSTRACT. The main purpose of this paper is to classify the compact simply connected Riemannian 4-symmetric spaces. As homogeneous manifolds, these spaces are of the form G/L where G is a connected compact semisimple Lie group with an automorphism σ of order four whose fixed point set is (essentially) L. Geometrically, they can be regarded as fiber bundles over Riemannian 2-symmetric spaces with totally geodesic fibers isometric to a Riemannian 2-symmetric space. A detailed description of these fibrations is also given. A compact simply connected Riemannian 4-symmetric space decomposes as a product $M_1 \times \cdots \times M_r$ where each irreducible factor is: (i) a Riemannian 2-symmetric space, (ii) a space of the form $\{U \times U \times U \times U\}/\Delta U$ with U a compact simply connected simple Lie group, $\Delta U =$ diagonal inclusion of U, (iii) $\{U \times U\}/\Delta U^\theta$ with U as in (ii) and U^θ the fixed point set of an involution θ of U, and (iv) U/K with U as in (ii) and K the fixed point set of an automorphism of order four of U. The core of the paper is the classification of the spaces in (iv). This is accomplished by first classifying the pairs $(\mathfrak{g}, \sigma)$ with $\mathfrak{g}$ a compact simple Lie algebra and σ an automorphism of order four of $\mathfrak{g}$. Tables are drawn listing all the possibilities for both the Lie algebras and the corresponding spaces. For U "classical," the automorphisms σ are explicitly constructed using their matrix representations. The idea of duality for 2-symmetric spaces is extended to 4-symmetric spaces and the duals are determined. Finally, those spaces that admit invariant almost complex structures are also determined: they are the spaces whose factors belong to the class (iv) with K the centralizer of a torus.

1. Introduction. In 1967, A. J. Ledger [13] initiated the study of generalized Riemannian symmetric spaces. These spaces are Riemannian manifolds (M, g) which admit at each point p in M an isometry s_p that has p as an isolated fixed point. The definition of these spaces arises as a natural extension of that of the symmetric spaces of Cartan. In fact, these spaces are also homogeneous [14]. Furthermore, if a regularity condition (trivially satisfied for ordinary symmetric spaces) is imposed on the isometries (s_p), then they can be chosen to have the same order n [9]. In this case, the spaces are said to be Riemannian regular n-symmetric spaces.

As homogeneous manifolds, the structure of Riemannian regular n-symmetric spaces is closely related to the study of finite order automorphisms of Lie groups. In this direction, the works of V. Kač (see e.g. [4, Chapter X]) and of J. A. Wolf and A. Gray [20] are of fundamental importance. In fact, in [20], a complete

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classification of the regular 3-symmetric spaces of semisimple type is obtained (see also [**3**]).

In this paper we classify the compact simply connected Riemannian regular 4-symmetric spaces. Geometrically, these spaces can be regarded as fiber bundles over Riemannian 2-symmetric spaces with totally geodesic fibers isometric to Riemannian 2-symmetric spaces (a result essentially due to R. Hermann [**5**, **6**], see §2 below). These fibrations will be described along with the classification.

The organization of the paper is as follows:

In §2, we establish the basic results needed for the rest of the paper. §3 (the heart of the paper) contains the classification, per se, of the automorphisms of order four of compact (complex) semisimple Lie algebras. The classification of the inner automorphisms of order four (Theorem 3.2) is fashioned after J. A. Wolf and A. Gray [**20**, I], and the classification of the outer automorphisms of order four (Theorem 3.3) is obtained directly from V. Kač's work [**4**, Chapter X].

§4 contains a description of the automorphisms of order four of the "classical" Lie algebras. Also, the idea of duality for 2-symmetric spaces is extended to 4-symmetric spaces (cf. [**9**, pp. 106–107]), and we obtain both, the compact simply connected Riemannian 4-symmetric spaces associated with these Lie algebras, and their corresponding (noncompact) dual spaces. This section is modeled after [**4**, Chapter X].

§5 is a synthesis of the previous ones, giving the classification of the compact simply connected Riemannian regular 4-symmetric spaces along with their fibrations (Theorems 5.2 and 5.4).

§6 is concerned with the classification of the compact almost Hermitian regular 4-symmetric spaces (Theorem 6.1). We also show that in addition to the Hermitian 2-symmetric spaces, a large class of almost Hermitian regular 3-symmetric spaces are almost Hermitian regular 4-symmetric spaces as well. The section concludes with the result that the manifold $M = \mathbf{SO}(2n+2)/\mathbf{U}(1) \times \mathbf{U}(n)$ can be made into an almost Hermitian regular 4-symmetric space in two nonequivalent ways; a result which is in contrast with 2- and 3-symmetric spaces.

ACKNOWLEDGMENTS. The author wishes to thank his supervisor Professor T. J. Willmore for introducing him into the study of Riemannian geometry. Parts of this paper appear already in the author's thesis [**7**]. Thanks are due to Professors A. Gray and O. Kowalski for their valuable suggestions during the preparation of this paper. A special word of thanks goes to the referee for his many detailed suggestions to improve the presentation of this paper. In particular the references to R. Hermann's work and [**17**, **18**] are his, as well as the strengthened version of Theorem 6.1 (which originally did not include the global separation of the Euclidean part).

2. Preliminaries. In this paper we shall be exclusively concerned with Riemannian regular n-symmetric spaces. Thus we shall include the regularity condition as part of the formal definition of these spaces, and we shall refer to them simply as Riemannian n-symmetric spaces.

DEFINITION. A Riemannian n-symmetric space is a connected C^∞-Riemannian manifold M which admits a family of isometries (s_x), x in M, with the following properties:

(i) For each x in M, the isometry s_x is of order n and has x as an isolated fixed point.

(ii) (Regularity condition.) For any two points x and y in M, the symmetries s_x and s_y satisfy

$$s_x \circ s_y = s_p \circ s_x$$

where $p = s_x(y)$.

The isometry s_x will normally be called the symmetry at x.

The homogeneous structure of n-symmetric spaces can be described as follows (see e.g. [**11** and **14**]): Let $I(M)$ denote the full isometry group of M, and let G be the identity component of the closed subgroup of $I(M)$ generated by the symmetries (s_x), x in M. Then G acts transitively on M and we have

$$M = G/L \quad \text{with } (G^\sigma)_0 \subset L \subset G^\sigma \tag{2.1}$$

where L is the isotropy group of G at a fixed point 0 in M, and σ is the automorphism (of order n) of G induced by conjugation with respect to s_0. G^σ, as usual, denotes the fixed point set of σ, and $(G^\sigma)_0$ denotes its identity component. Let $\mathfrak{g}$ be the Lie algebra of G, and σ (same letter) the automorphism of $\mathfrak{g}$ induced by σ. Since L is compact, G/L is reductive, and $\mathfrak{g}$ admits an $\mathrm{Ad}(L)$- and σ-invariant decomposition $\mathfrak{g} = \mathfrak{l} + \mathfrak{m}$ (vector space direct sum) where $\mathfrak{l} = \mathfrak{g}^\sigma$ is the Lie algebra corresponding to L, and $\mathfrak{m}$ can be identified with the tangent space of G/L at L, thus $\mathfrak{m}$ comes equipped with an $\mathrm{Ad}(L)$- and σ-invariant inner product. And conversely (see e.g. [**14**]) given a connected Lie group G and an automorphism σ of order n of G, and a subgroup L that satisfies (2.1), assume that $\mathfrak{g}$ admits an $\mathrm{Ad}(L)$- and σ-invariant decomposition $\mathfrak{g} = \mathfrak{l} + \mathfrak{m}$ and $\mathfrak{m}$ admits an $\mathrm{Ad}(L)$- and σ-invariant inner product $\langle\ ,\ \rangle$, then G/L can be made into a Riemannian n-symmetric space. It follows from this that the problem of classification of compact simply connected Riemannian n-symmetric spaces is equivalent to the problem of classifying automorphisms of order n of compact semisimple Lie algebras.

In the special case when $n = 4$, an $\mathrm{Ad}(L)$- and σ-invariant decomposition of $\mathfrak{g}$ is readily available; namely

$$\mathfrak{g} = \mathfrak{l} + \mathfrak{v} + \mathfrak{p} \qquad \text{(vector space direct sum)} \tag{2.2}$$

where, as above, $\mathfrak{l} = \mathfrak{g}^\sigma$, and $\mathfrak{m} = \mathfrak{v} + \mathfrak{p}$ with $\mathfrak{v}$ the eigenspace of σ for the eigenvalue -1, and $\mathfrak{p}$ the eigenspace of σ^2 for the eigenvalue -1. It follows, in particular, that $\mathfrak{v}$ and $\mathfrak{p}$ define an almost-product structure on G/L, and this brings us to consider the fibrations of Riemannian 4-symmetric spaces.

Let M be a compact Riemannian 4-symmetric space, and let (s_x), x in M, be its symmetries. At each point x in M let F_x denote the connected component through x of the fixed point set of s_x^2. Two options are available: either $F_x = \{x\}$ or $\dim F_x > 0$. In the first case s_x^2 is the usual geodesic involution and M is a 2-symmetric space, in fact, a Hermitian 2-symmetric space. In the second case, F_x is a complete totally geodesic submanifold [**6**, Volume II, p. 61], and (F_x), x in M, determines a foliation of M. Let $B = M/F$ denote the set of leaves of the foliation, and let $\pi\colon M \to B$ be the projection $x \to (F_x)$, for x in M. Then (R. Hermann [**5**, **6**]) π defines a locally trivial fiber bundle. Furthermore, since for each $y \in F_x$ the symmetry s_y preserves F_x, the fibers are 2-symmetric spaces, and it is not difficult to see that the base space B is also a 2-symmetric space. (Of course, in the notation of (2.2), the (involutive) distribution defined by $\mathfrak{v}$ coincides with the tangent spaces to the leaves, and $\mathfrak{p}$ defines the orthogonal complementary distribution.) A detailed description of these fibrations will be given along with the classification of the spaces (see Tables III, IV and V in §5).

3. Automorphisms of order four of compact semisimple Lie algebras. In this section we set about classifying the pairs $(\mathfrak{g}, \sigma)$ with $\mathfrak{g}$ a compact semisimple Lie algebra and σ an automorphism of order four. Finite order automorphisms of semisimple Lie algebras have been extensively studied, and there is, indeed, a well-defined path that can be followed to obtain their classification. Here we shall follow the works of J. Wolf and A. Gray [**20**], and of V. Kač [**4**, Chapter X]. Details will be reduced to a minimum and whenever possible we shall refer to the two mentioned works for any further explanations.

The section starts with a description of the three different types of irreducible factors into which a pair $(\mathfrak{g}, \sigma)$ can be decomposed. An immediate consequence is that the classification reduces essentially to classifying the pairs $(\mathfrak{g}, \sigma)$ with $\mathfrak{g}$ simple and σ an automorphism of order four (Type III). The classification of such pairs is naturally divided into two parts depending on whether σ is an inner automorphism or an outer automorphism. Theorems 3.2 and 3.3 contain, respectively, the classification for each case.

PROPOSITION 3.1 (J. WOLF AND A. GRAY [**20**, P. 106]). *Let $(\mathfrak{g}, \sigma)$ be a pair with $\mathfrak{g}$ a compact semisimple Lie algebra and σ an automorphism of order four such that $\mathfrak{g}^\sigma$ does not contain any proper ideals of $\mathfrak{g}$. Then $\mathfrak{g}$ splits into a σ-invariant direct sum of ideals $\mathfrak{g} = \mathfrak{g}_1 + \cdots + \mathfrak{g}_r$ such that each pair $(\mathfrak{g}_i, \sigma_i)$ with σ_i the restriction of σ to $\mathfrak{g}_i$ is isomorphic to one of the following types:*

(I) $(\mathfrak{l}, \lambda)$, *where the compact Lie algebra $\mathfrak{l}$ is the direct sum $\mathfrak{l} = \mathfrak{l}_1 + \mathfrak{l}_1 + \mathfrak{l}_1 + \mathfrak{l}_1$ of simple ideals and*

$$\lambda(X_1, X_2, X_3, X_4) = (X_4, X_1, X_2, X_3),$$

for any $X_i \in \mathfrak{l}_1$, $i = 1, 2, 3, 4$.

(II) $(\mathfrak{l}, \lambda)$, *where the compact Lie algebra $\mathfrak{l}$ is the direct sum $\mathfrak{l} = \mathfrak{l}_1 + \mathfrak{l}_1$ of simple ideals and $\lambda(X_1, X_2) = (\theta X_2, X_1)$ where θ is either the identity or an involutive automorphism of $\mathfrak{l}_1$.*

(III) $(\mathfrak{l}, \lambda)$, *where $\mathfrak{l}$ is a compact simple Lie algebra and σ is an automorphism of order two or four of $\mathfrak{l}$.*

The classification of the pairs of Type I, Type II, and Type III with λ an involution, is readily available. The core of this section is the classification of the pairs of Type III with λ of order four. This is the content of the next two theorems.

THEOREM 3.2. *Let $\tilde{\sigma}$ be an inner automorphism of order four on a compact or complex simple Lie algebra $\mathfrak{g}$. Choose a Cartan subalgebra and let $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ be a simple rootsystem for $\mathfrak{g}$. Then $\tilde{\sigma}$ is conjugate in the full automorphism group of $\mathfrak{g}$ to some $\sigma = \mathrm{Ad}(\exp 2\pi\sqrt{-1}X)$, where a complete list of the possibilities for X along with the fixed point set $\mathfrak{g}^\sigma$ and a simple root system Δ_X of $\mathfrak{g}^\sigma$ is given in Table* I.

TABLE I. Inner Automorphisms of Order Four of Compact or Complex Simple Lie Algebras

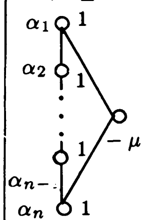
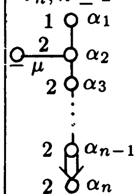
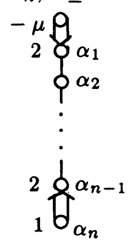
$\mathfrak{g}$	X	Δ_X	$\mathfrak{g}^\sigma$
$\mathfrak{a}_1$	$\frac{1}{4}V_1$	empty	$\mathfrak{t}^1$
$\mathfrak{a}_n, n \geq 2$ 	$\frac{1}{4}V_i$ $1 \leq i \leq [\frac{1}{2}(n+1)]$	$\{\alpha_1, \ldots, \alpha_{i-1}; \alpha_{i+1}, \ldots, \alpha_n\}$	$\mathfrak{a}_{i-1} \oplus \mathfrak{a}_{n-i} \oplus \mathfrak{t}^1$
	$\frac{1}{4}(V_i + V_j)$ $1 \leq i < j \leq n$ $i-1 \leq j-i-1 \leq n-j$	$\{\alpha_1, \ldots, \alpha_{i-1}; \alpha_{i+1}, \ldots, \alpha_{j-1};$ $\alpha_{j+1}, \ldots, \alpha_n\}$	$\mathfrak{a}_{i-1} \oplus \mathfrak{a}_{j-i-1} \oplus \mathfrak{a}_{n-j} \oplus \mathfrak{t}^2$
	$\frac{1}{4}(V_i + V_j + V_k)$ $1 \leq i < j < k \leq n$ $i-1 \leq j-i-1 \leq k-j-1 \leq n-k$	$\{\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_{j-1},$ $\alpha_{j+1}, \ldots, \alpha_{k-1}; \alpha_{k+1}, \ldots, \alpha_n\}$	$\mathfrak{a}_{i-1} \oplus \mathfrak{a}_{j-i-1} \oplus \mathfrak{a}_{k-j-1}$ $\oplus \mathfrak{a}_{n-k} \oplus \mathfrak{t}^3$
$\mathfrak{b}_n, n \geq 2$ 	$\frac{1}{4}V_1$	$\{\alpha_2, \ldots, \alpha_n\}$	$\mathfrak{b}_{n-1} \oplus \mathfrak{t}^1$
	$\frac{1}{2}V_i$ $2 \leq i \leq n$	$\{\alpha_1, \ldots, \alpha_{i-1}; \alpha_{i+1}, \ldots, \alpha_n\}$	$\mathfrak{a}_{i-1} \oplus \mathfrak{b}_{b-i} \oplus \mathfrak{t}^1$
	$\frac{1}{4}(V_1 + 2V_i)$ $2 \leq i \leq n$	$\{\alpha_2, \ldots, \alpha_{i-1}; \alpha_{i+1}, \ldots, \alpha_n\}$	$\mathfrak{a}_{i-2} \oplus \mathfrak{b}_{n-i} \oplus \mathfrak{t}^2$
	$\frac{1}{2}(V_i + V_j)$ $2 \leq i < j \leq n$	$\{\alpha_1, \ldots, \alpha_{i-1}; \alpha_{i+1}, \ldots, \alpha_{j-1};$ $\alpha_{j+1}, \ldots, \alpha_n\} \cup \{-\mu\}$	$\mathfrak{d}_i \oplus \mathfrak{a}_{j-i-1} \oplus \mathfrak{b}_{n-j} \oplus \mathfrak{t}^1$
$\mathfrak{c}_n, n \geq 3$ 	$\frac{1}{4}V_n$	$\{\alpha_1, \ldots, \alpha_{n-1}\}$	$\mathfrak{a}_{n-1} \oplus \mathfrak{t}^1$
	$\frac{1}{2}V_i$ $1 \leq i \leq n-1$	$\{\alpha_1, \ldots, \alpha_{i-1}; \alpha_{i+1}, \ldots, \alpha_n\}$	$\mathfrak{a}_{i-1} \oplus \mathfrak{c}_{n-i} \oplus \mathfrak{t}^1$
	$\frac{1}{4}(V_n + 2V_i)$ $1 \leq i \leq [\frac{1}{2}n]$	$\{\alpha_1, \ldots, \alpha_{i-1}; \alpha_{i+1}, \ldots, \alpha_{n-1}\}$	$\mathfrak{a}_{i-1} \oplus \mathfrak{a}_{n-i-1} \oplus \mathfrak{t}^2$
	$\frac{1}{2}(V_i + V_j)$ $1 \leq i < j \leq n-1$ $i \leq n-j$	$\{\alpha_1, \ldots, \alpha_{i-1}; \alpha_{i+1}, \ldots, \alpha_{j-1};$ $\alpha_{j+1}, \ldots, \alpha_n\} \cup \{-\mu\}$	$\mathfrak{c}_i \oplus \mathfrak{a}_{j-i-1} \oplus \mathfrak{c}_{n-j} \oplus \mathfrak{t}^1$

TABLE I (CONTINUED)

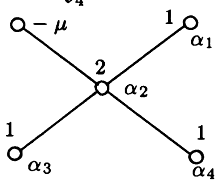
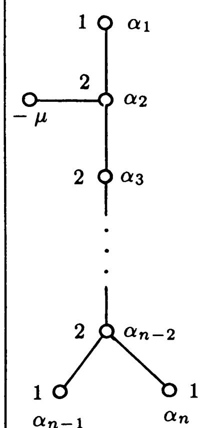
$\mathfrak{g}$	X	Δ_X	$\mathfrak{g}^\sigma$
$\mathfrak{d}_4$ 	$\frac{1}{4}V_1$	$\{\alpha_2, \alpha_3, \alpha_4\}$	$\mathfrak{a}_3 \oplus \mathfrak{t}^1$
	$\frac{1}{4}(V_1 + V_4)$	$\{\alpha_2, \alpha_3\}$	$\mathfrak{a}_2 \oplus \mathfrak{t}^2$
	$\frac{1}{2}V_2$	$\{\alpha_1, \alpha_3, \alpha_4\}$	$\mathfrak{a}_1 \oplus \mathfrak{a}_1 \oplus \mathfrak{a}_1 \oplus \mathfrak{t}^1$
	$\frac{1}{4}(V_1 + V_3 + V_4)$	$\{\alpha_2\}$	$\mathfrak{a}_1 \oplus \mathfrak{t}^3$
	$\frac{1}{4}(V_1 + 2V_2)$	$\{\alpha_3, \alpha_4\}$	$\mathfrak{a}_1 \oplus \mathfrak{a}_1 \oplus \mathfrak{t}^2$
$\mathfrak{d}_n, n > 4$ 	$\frac{1}{4}V_1$	$\{\alpha_2, \dots, \alpha_n\}$	$\mathfrak{d}_{n-1} \oplus \mathfrak{t}^1$
	$\frac{1}{4}V_n$	$\{\alpha_1, \dots, \alpha_{n-1}\}$	$\mathfrak{a}_{n-1} \oplus \mathfrak{t}^1$
	$\frac{1}{4}(V_1 + V_n)$	$\{\alpha_2, \dots, \alpha_{n-1}\}$	$\mathfrak{a}_{n-2} \oplus \mathfrak{t}^2$
	$\frac{1}{4}(V_{n-1} + V_n)$	$\{\alpha_1, \dots, \alpha_{n-2}\}$	$\mathfrak{a}_{n-2} \oplus \mathfrak{t}^2$
	$\frac{1}{2}V_i$ $2 \le i \le n-2$	$\{\alpha_1, \dots, \alpha_{i-1}; \alpha_{i+1}, \dots, \alpha_n\}$	$\mathfrak{a}_{i-1} \oplus \mathfrak{d}_{n-i} \oplus \mathfrak{t}^1$
	$\frac{1}{4}(V_1 + V_{n-1} + V_n)$	$\{\alpha_2, \dots, \alpha_{n-2}\}$	$\mathfrak{a}_{n-3} \oplus \mathfrak{t}^3$
	$\frac{1}{4}(V_1 + 2V_i)$ $2 \le i \le n-2$	$\{\alpha_2, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_n\}$	$\mathfrak{a}_{i-2} \oplus \mathfrak{d}_{n-i} \oplus \mathfrak{t}^2$
	$\frac{1}{4}(V_n + 2V_i)$ $2 \le i \le [n/2]$	$\{\alpha_1, \dots, \alpha_{i-1};\ \alpha_{i+1}, \dots, \alpha_{n-1}\}$	$\mathfrak{a}_{i-2} \oplus \mathfrak{a}_{n-i-1} \oplus \mathfrak{t}^2$
	$\frac{1}{2}(V_i + V_j)$ $2 \le i < j \le n-2$ $i \le n-j$	$\{\alpha_1, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_{j-1},$ $\alpha_{j+1}, \dots, \alpha_n\} \cup \{-\mu\}$	$\mathfrak{d}_i \oplus \mathfrak{a}_{j-i-1} \oplus \mathfrak{d}_{n-j} \oplus \mathfrak{t}^1$

TABLE I (CONTINUED)

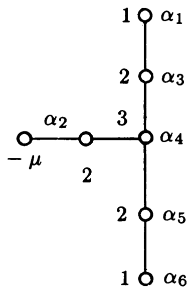
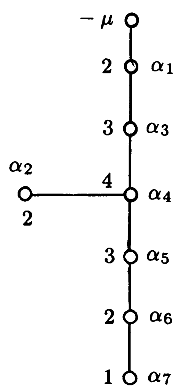
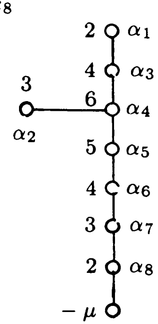
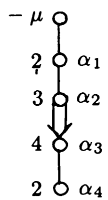
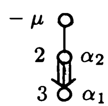
$\mathfrak{g}$	X	Δ_X	$\mathfrak{g}^\sigma$
$\mathfrak{e}_6$ 	$\frac{1}{4}V_1$	$\{\alpha_2, \ldots, \alpha_6\}$	$\mathfrak{d}_5 \oplus \mathfrak{t}^1$
	$\frac{1}{4}(V_1 + V_6)$	$\{\alpha_2, \ldots, \alpha_5\}$	$\mathfrak{d}_4 \oplus \mathfrak{t}^2$
	$\frac{1}{2}V_2$	$\{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$	$\mathfrak{a}_5 \oplus \mathfrak{t}^1$
	$\frac{1}{2}V_3$	$\{\alpha_1, \alpha_2, \alpha_4, \alpha_5, \alpha_6\}$	$\mathfrak{a}_4 \oplus \mathfrak{a}_1 \oplus \mathfrak{t}^1$
	$\frac{1}{4}(V_1 + 2V_5)$	$\{\alpha_2, \alpha_3, \alpha_4, \alpha_6\}$	$\mathfrak{a}_3 \oplus \mathfrak{a}_1 \oplus \mathfrak{t}^2$
	$\frac{1}{4}(V_1 + 2V_2)$	$\{\alpha_3, \alpha_4, \alpha_5, \alpha_6\}$	$\mathfrak{a}_4 \oplus \mathfrak{t}^2$
	$\frac{3}{4}V_4$	$\{\alpha_1, \alpha_2, \alpha_3, \alpha_5, \alpha_6\}$	$\mathfrak{a}_2 \oplus \mathfrak{a}_1 \oplus \mathfrak{a}_2 \oplus \mathfrak{t}^1$
	$\frac{1}{2}(V_3 + V_5)$	$\{\alpha_1, \alpha_2, \alpha_4, \alpha_6\} \cup \{-\mu\}$	$\mathfrak{a}_1 \oplus \mathfrak{a}_3 \oplus \mathfrak{a}_1 \oplus \mathfrak{t}^1$
$\mathfrak{e}_7$ 	$\frac{1}{4}V_7$	$\{\alpha_1, \ldots, \alpha_6\}$	$\mathfrak{e}_6 \oplus \mathfrak{t}^1$
	$\frac{1}{2}V_1$	$\{\alpha_2, \ldots, \alpha_7\}$	$\mathfrak{d}_6 \oplus \mathfrak{t}^1$
	$\frac{1}{2}V_6$	$\{\alpha_1, \ldots, \alpha_5, \alpha_7\}$	$\mathfrak{d}_5 \oplus \mathfrak{a}_1 \oplus \mathfrak{t}^1$
	$\frac{1}{2}V_2$	$\{\alpha_1, \alpha_3, \ldots, \alpha_7\}$	$\mathfrak{a}_6 \oplus \mathfrak{t}^1$
	$\frac{1}{4}(V_7 + 2V_1)$	$\{\alpha_2, \ldots, \alpha_6\}$	$\mathfrak{d}_5 \oplus \mathfrak{t}^2$
	$\frac{1}{4}(V_7 + 2V_2)$	$\{\alpha_1, \alpha_3, \ldots, \alpha_6\}$	$\mathfrak{a}_5 \oplus \mathfrak{t}^2$
	$\frac{3}{4}V_3$	$\{\alpha_1, \alpha_2, \alpha_4, \ldots, \alpha_6, \alpha_7\}$	$\mathfrak{a}_1 \oplus \mathfrak{a}_5 \oplus \mathfrak{t}^1$
	$\frac{3}{4}V_5$	$\{\alpha_1, \ldots, \alpha_4, \alpha_6, \alpha_7\}$	$\mathfrak{a}_4 \oplus \mathfrak{a}_2 \oplus \mathfrak{t}^1$
	$\frac{1}{2}(V_1 + V_6)$	$\{\alpha_2, \ldots, \alpha_5, \alpha_7\} \cup \{-\mu\}$	$\mathfrak{a}_1 \oplus \mathfrak{a}_1 \oplus \mathfrak{d}_4 \oplus \mathfrak{t}^1$
	$\frac{1}{2}(V_1 + V_2)$	$\{\alpha_3, \ldots, \alpha_7\} \cup \{-\mu\}$	$\mathfrak{a}_5 \oplus \mathfrak{a}_1 \oplus \mathfrak{t}^1$
	V_4	$\{\alpha_1, \alpha_2, \alpha_3, \alpha_5, \alpha_6, \alpha_7\} \cup \{-\mu\}$	$\mathfrak{a}_3 \oplus \mathfrak{a}_3 \oplus \mathfrak{a}_1$

TABLE I (CONTINUED)

$\mathfrak{g}$	X	Δ_X	$\mathfrak{g}^\sigma$
$\mathfrak{e}_8$ 	$\frac{1}{2}V_1$	$\{\alpha_2, \ldots, \alpha_8\}$	$\mathfrak{d}_7 \oplus \mathfrak{t}^1$
	$\frac{1}{2}V_8$	$\{\alpha_1, \ldots, \alpha_7\}$	$\mathfrak{e}_7 \oplus \mathfrak{t}^1$
	$\frac{3}{4}V_2$	$\{\alpha_1, \alpha_3, \ldots, \alpha_8\}$	$\mathfrak{a}_7 \oplus \mathfrak{t}^1$
	$\frac{3}{4}V_7$	$\{\alpha_1, \ldots, \alpha_6, \alpha_8\}$	$\mathfrak{e}_6 \oplus \mathfrak{a}_1 \oplus \mathfrak{t}^1$
	$\frac{1}{2}(V_1 + V_8)$	$\{\alpha_2, \ldots, \alpha_7\} \cup \{-\mu\}$	$\mathfrak{d}_6 \oplus \mathfrak{a}_1 \oplus \mathfrak{t}^1$
	V_3	$\{\alpha_1, \alpha_2, \alpha_4, \ldots, \alpha_8\} \cup \{-\mu\}$	$\mathfrak{a}_1 \oplus \mathfrak{a}_7$
	V_6	$\{\alpha_1, \ldots, \alpha_5, \alpha_7, \alpha_8\} \cup \{-\mu\}$	$\mathfrak{d}_5 \oplus \mathfrak{a}_3$
$\mathfrak{f}_4$ 	$\frac{1}{2}V_1$	$\{\alpha_2, \alpha_3, \alpha_4\}$	$\mathfrak{c}_3 \oplus \mathfrak{t}^1$
	$\frac{1}{2}V_4$	$\{\alpha_1, \alpha_2, \alpha_3\}$	$\mathfrak{b}_3 \oplus \mathfrak{t}^1$
	$\frac{3}{4}V_2$	$\{\alpha_1, \alpha_3, \alpha_4\}$	$\mathfrak{a}_1 \oplus \mathfrak{a}_2 \oplus \mathfrak{t}^1$
	$\frac{1}{2}(V_1 + V_4)$	$\{\alpha_2, \alpha_3\} \cup \{-\mu\}$	$\mathfrak{a}_1 \oplus \mathfrak{b}_2 \oplus \mathfrak{t}^1$
	V_3	$\{\alpha_1, \alpha_2, \alpha_4\} \cup \{-\mu\}$	$\mathfrak{a}_3 \oplus \mathfrak{a}_1$
$\mathfrak{g}_2$ 	$\frac{1}{2}V_2$	$\{\alpha_1\}$	$\mathfrak{a}_1 \oplus \mathfrak{t}^1$
	$\frac{3}{4}V_1$	$\{\alpha_2\}$	$\mathfrak{a}_1 \oplus \mathfrak{t}^1$

THEOREM 3.3. *Let σ be an outer automorphism of order four on a compact (or complex) simple Lie algebra $\mathfrak{g}$, and let $\mathfrak{g}^\sigma$ denote the fixed point set of σ. Then $\mathfrak{g}$ is of type $\mathfrak{a}_n$, $\mathfrak{d}_n$ or $\mathfrak{e}_6$ and a complete list of all the possibilities for $\mathfrak{g}^\sigma$ is given in Table* II.

TABLE II. Outer Automorphisms of Order Four of Compact or Complex Simple Lie algebras

$\mathfrak{g}^\sigma$ semisimple		(dim center $\mathfrak{g}^\sigma = 1$)	
$\mathfrak{g}$	$\mathfrak{g}^\sigma$	$\mathfrak{g}$	$\mathfrak{g}^\sigma$
$\mathfrak{a}_2$	$\mathfrak{a}_1$	$\mathfrak{a}_{2n-1}$	$\mathfrak{c}_{n-1} \oplus \mathfrak{t}^1$
$\mathfrak{a}_{2n}$ $n > 1$	$\mathfrak{c}_j \oplus \mathfrak{b}_{n-j}$ $1 \leq j \leq n$	$n > 2$	$\mathfrak{a}_{n-1} \oplus \mathfrak{t}^1$
$\mathfrak{a}_{2n-1}$ $n > 2$	$\mathfrak{d}_i \oplus \mathfrak{c}_{n-i}$ $2 \leq i \leq n-1$	$\mathfrak{d}_{n+1}$ $n > 1$	$\mathfrak{b}_{i-1} \oplus \mathfrak{a}_{n-i-j} \oplus \mathfrak{b}_j \oplus \mathfrak{t}^1$ $1 \leq i \leq \frac{1}{2}[n+1],\ 0 \leq j \leq n-i$
$\mathfrak{e}_6$	$\mathfrak{a}_3 \oplus \mathfrak{a}_1$ $\mathfrak{b}_3 \oplus \mathfrak{b}_1$	$\mathfrak{e}_6$	$\mathfrak{c}_3 \oplus \mathfrak{t}^1$

COROLLARY 3.4. *Two outer automorphisms of order four of a compact (or complex) simple Lie algebra $\mathfrak{g}$ are conjugate in* $\mathrm{Aut}(\mathfrak{g})$ *if and only if their fixed point Lie algebras are isomorphic.*

This corollary is no longer true for inner automorphisms.

An inspection into the classification tables yields the following

COROLLARY 3.5. *Let $\mathfrak{g}$ be a compact or complex simple Lie algebra of type $\mathfrak{a}_n$, $\mathfrak{f}_4$, $\mathfrak{e}_6$ or $\mathfrak{e}_8$. Then two automorphisms of order four of $\mathfrak{g}$ are conjugate in* $\mathrm{Aut}(\mathfrak{g})$ *if and only if their fixed point Lie algebras are isomorphic.*

We now proceed to outline the proofs of Theorems 3.2 and 3.3.

Theorem 3.2 is fashioned after J. Wolf and A. Gray [**20**, §§2, 3]. However, a slight modification is made by also using V. Kač's method [**4**, Chapter X, §5]. On one hand, the latter method has the advantage of rendering the classification up to conjugation within the full group of automorphisms. On the other hand, the former method gives a very explicit description of the automorphisms. This description will in turn be used to classify the almost Hermitian 4-symmetric spaces in §6.

The classification of the outer automorphisms (Theorem 3.3) will be obtained following the method of V. Kač (loc. cit.).

Notation 3.6. In what follows $\mathfrak{g}$ denotes a compact simple Lie algebra and G the compact centerless Lie group whose Lie algebra is $\mathfrak{g}$. Ad and ad denote the adjoint representation of G and $\mathfrak{g}$ respectively and $\exp\colon \mathfrak{g} \to G$ is the exponential map. Let T be a maximal torus of G and $\mathfrak{t}_0$ the corresponding subalgebra of $\mathfrak{g}$. Then the complex subalgebra $\mathfrak{t}$ generated by $\mathfrak{t}_0$ in $\mathfrak{g}^{\mathbf{C}}$, $\mathfrak{g}^{\mathbf{C}}$ the complexification of $\mathfrak{g}$, is a Cartan subalgebra whose root system will be denoted by Δ. Furthermore, we fix a system $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ of simple roots and denote by $\mu = \sum m_i \alpha_i$ the corresponding maximal root. Since the roots have pure imaginary values on $\mathfrak{t}_0$, a simplex $\mathcal{D}_0$ in $\sqrt{-1}\mathfrak{t}_0$ can be defined as follows:

$$\mathcal{D}_0 = \{X \in \mathfrak{t}_0 \colon \alpha_i(X) \geq 0,\ i = 1, \ldots, n,\ \mu(X) \leq 1\}.$$

$\mathcal{D}_0$ has vertices $\{V_0, V_1, \ldots, V_n\}$ given by $V_0 = 0$, $\alpha_i(V_j) = \delta_{ij}/m_i$.

$\mathcal{D}_0$ has the property that every element in G is conjugate to an element in $\exp(2\pi\sqrt{-1}\mathcal{D}_0)$. As a consequence we have that any inner automorphism of $\mathfrak{g}$ is conjugate to an automorphism of the form

$$\sigma = \mathrm{Ad}(g) \quad \text{where } g = \exp(2\pi\sqrt{-1}X) \text{ with } X \in \mathcal{D}_0. \tag{3.6.1}$$

Normalization 3.7. Let $\sigma = \mathrm{Ad}(g)$ be as in (3.6.1) and assume that σ is of order four. Then $\alpha_i(4X) = n_i \in \mathbf{Z}$ for all i. Since $X \in \mathcal{D}_0$, it can be decomposed as $X = \sum c_i V_i$, and we have that $n_i = \alpha_i(4X) = 4c_i/m_i$, hence

$$X = \sum (n_i m_i/4) V_i \quad \text{with } n_i = \alpha_i(4X) \in \mathbf{Z}.$$

X can be replaced by a transform $\omega(X) + \gamma$ with ω an element of the Weyl group of $\mathfrak{g}$ with respect to $\mathfrak{t}_0$ and γ an element of $(2\pi\sqrt{-1})^{-1}\mathfrak{t}_e$ where $\mathfrak{t}_e = \{H \in \mathfrak{t}_0 \colon \exp H = e\}$ is the unit lattice. This transform can be made in such a way as to minimize $\sum n_i m_i$. Then we have the following normalization (see [**20**, Proposition 2.7]):

(i) $0 \leq n_i < 4$ and $0 < \sum n_i m_i \leq 4$, and $\sum n_i m_i = 4$ implies that $m_j > 1$ whenever $n_j \neq 0$,

(ii) $n_j \leq 2$ if $m_j = 1$,

(iii) the sets $I_t = \{i \colon n_i \geq t\}$ have cardinality $|I_1| \geq 1$, $|I_2| \leq 1$, and I_t is empty for $t > 2$,

(iv) $\alpha_i(2X) \notin \mathbf{Z}$ for some i and $n_i = \alpha_i(4X) \in \mathbf{Z}$ for all i.

The last condition simply states that σ is of order four.

The next proposition lists all the possibilities for $X = \sum (n_i m_i/4) V_i$ normalized according to the above conditions (i)–(iv).

PROPOSITION 3.8. *Let ϕ be an inner automorphism of order four on a compact or complex simple Lie algebra $\mathfrak{g}$. Then ϕ is conjugate, by an inner automorphism of $\mathfrak{g}$, to an automorphism of the form $\sigma = \mathrm{Ad}(\exp 2\pi\sqrt{-1}X)$ where $X = \sum (n_i m_i/4) V_i$ is an element of $\mathcal{D}_0$ as given below:*

(i) *$X = \frac{1}{4} V_i$ with $m_i = 1$, $n_i = 1$,*

(ii) *$X = \frac{1}{4}(V_i + V_j)$ with $m_i = m_j = 1$, $n_i = n_j = 1$, $X = \frac{1}{2} V_i$ with $m_i = 2$, $n_i = 1$,*

(iii) *$X = \frac{1}{4}(V_i + V_j + V_k)$ with $m_i = m_j = m_k = 1$, $n_i = n_j = n_k = 1$, $X = \frac{1}{4}(2V_i + V_j)$ with $m_i = m_j = 1$, $n_i = 2$, $n_j = 1$, $X = \frac{1}{4}(2V_i + V_j)$ with $m_i = 2$, $m_j = 1$, $n_i = n_j = 1$, $X = \frac{3}{4} V_j$ with $m_j = 3$, $n_j = 1$,*

(iv) *$X = \frac{1}{2}(V_i + V_j)$ with $m_i = m_j = 2$, $n_i = n_j = 1$, $X = V_i$ with $m_i = 4$, $n_i = 1$.*

The proof can safely be omitted. It is based on Normalization 3.7.

The above proposition gives us a way of classifying all inner automorphisms of order four of a compact simple Lie algebra, the classification being up to conjugation within the group of inner automorphisms. However, our final objective is to obtain this classification but up to conjugation within the full group of automorphisms. To accomplish this we need the following:

For $X = \sum(n_i m_i/4)V_i$ as in Proposition 3.8, set $m_0 = 1$, and let n_0 be the integer ≥ 0 such that $\sum_{i=0}^{n} n_i m_i = 4$. We have (see e.g. [**4**, Chapter X, Theorem 5.16]): Let $X, X' \in \mathcal{D}_0$ be as in Proposition 3.8, $X = \sum(n_i m_i/4)V_i$, $X' = \sum(n'_i m'_i/4)V_i$. Then the automorphisms $\sigma = \mathrm{Ad}(g)$ and $\sigma' = \mathrm{Ad}(g')$, with $g = \exp(2\pi\sqrt{-1}X)$ and $g' = \exp(2\pi\sqrt{-1}X')$, are conjugate within $\mathrm{Aut}(\mathfrak{g})$ if and only if the sequence $(n_0, n_1, \ldots, n_n)$ can be transformed into the sequence $(n'_0, n'_1, \ldots, n'_n)$ by an automorphism of the extended Dynkin diagram of $\mathfrak{g}$.

For X and X' as above, we shall write $X \sim X'$ to indicate that their respective sequences can be transformed one into the other by an automorphism of the extended Dynkin diagram of $\mathfrak{g}$. The above criterion is also contained in [**20**, Proposition 2.6].

As an immediate application we have the following

LEMMA 3.9. *Let $\mathfrak{g}$ be a complex simple Lie algebra. Let $X, Y \in \mathcal{D}_0$ be as in Proposition* 3.8, *$X = \frac{1}{4}(V_i + V_j)$ and $Y = \frac{1}{4}(2V_i + V_j)$ with $m_i = m_j = 1$, $i \neq j$. Then $\mathfrak{g}$ is of type $\mathfrak{a}_n$, $\mathfrak{d}_n$ or $\mathfrak{e}_6$ and $X \sim Y$.*

PROOF. As there are two distinct indices i, j with $m_i = m_j = 1$, $\mathfrak{g}$ must be of type $\mathfrak{a}_n$ $(n > 1)$, $\mathfrak{d}_n$ $(n > 3)$ or $\mathfrak{e}_6$.

If $\mathfrak{g}$ is of type $\mathfrak{e}_6$, then

$$m_1 = m_6 = 1 = m_0.$$

The sequence for X is $(2, 1, 0, \ldots, 0, 1)$, while for Y there are two possible sequences: $(1, 2, 0, \ldots, 0, 1)$ or $(1, 1, 0, \ldots, 0, 2)$. The symmetry of the extended Dynkin diagram of $\mathfrak{e}_6$ shows that anyone of these three sequences can be transformed into any one of the other two. (See Table I for a description of the extended Dynkin diagram of $\mathfrak{e}_6$.)

Similar arguments work for the other two cases.

As a consequence, when using Proposition 3.8 to obtain the classification, up to conjugation within the full group of automorphisms of $\mathfrak{g}$, the automorphisms induced by elements of the form $X = \frac{1}{4}(2V_i + V_j)$ with $m_i = m_j = 1$, $i \neq j$, can be omitted from the list.

It is now a simple matter to verify the lists in Table I.

To conclude the proof of Theorem 3.2, the following criterion can be used to determine a simple root system Δ_X for the fixed point Lie algebra $\mathfrak{g}^\sigma$ of $\sigma = \mathrm{Ad}(\exp 2\pi\sqrt{-1}X)$ with $X = \sum(n_i m_i/4)V_i$ as in Proposition 3.8 (cf. [**20**, Proposition 2.8]):

$$\Delta_X = \begin{cases} \{\alpha_i \in \Delta \colon n_i = 0\} & \text{if } \mu(X) < 1, \\ \{\alpha_i \in \Delta \colon n_i = 0\} \cup \{-\mu\} & \text{if } \mu(X) = 1. \end{cases}$$

The proof of Theorem 3.3 consists of a straightforward application of the work of V. Kač [**4**, Chapter X, §5]. Therefore it will be omitted.

4. The automorphisms of order four of the classical compact Lie algebra. The object of this section is to describe the automorphisms of order four of the "classical" Lie algebras $\mathfrak{su}(n)$, $\mathfrak{so}(n)$ and $\mathfrak{sp}(n)$. This is done in much the same vein as in [**4**, Chapter X, §2]. The main result is that at the same time we obtain the compact simply connected 4-symmetric spaces associated with these Lie algebras. In fact, a complete description of the fibrations of these spaces is given. We list for each one of them both the fiber and the base space. Furthermore, the idea of duality for 2-symmetric spaces is extended to 4-symmetric spaces. This extension is very natural and provides us with a large class of examples of noncompact 4-symmetric spaces. A description of these dual spaces will also be given (cf. [**12**, p. 106]).

Let $\mathfrak{u}$ be a compact simple Lie algebra and σ an automorphism of order four of $\mathfrak{u}$. Let $\mathfrak{u} = \mathfrak{l}_0 + \mathfrak{v}_0 + \mathfrak{p}_*$ be the direct sum decomposition of $\mathfrak{u}$ induced by σ as in (2.2). Then $\mathfrak{g}_0 = \mathfrak{l}_0 + \mathfrak{v}_0 + \sqrt{-1}\mathfrak{p}_*$ is a noncompact real form of the complexification, $\mathfrak{g} = \mathfrak{u}^{\mathbf{C}}$, of $\mathfrak{u}$. σ defines in a natural way an automorphism σ_0 of order four on $\mathfrak{g}_0$. The pair $(\mathfrak{g}_0, \sigma_0)$ is the dual of $(\mathfrak{u}, \sigma)$. Actually the pairs $(\mathfrak{g}_0, \sigma_0^2)$ and $(\mathfrak{u}, \sigma^2)$ are duals one of each other in the ordinary sense of 2-symmetric spaces.

Here we adopt the same notation as in [**4**, Chapter X, §2] for the classical groups and their Lie algebras.

The unit matrix of order n will be denoted by I_n, and we put

$$I_{p,q} = \begin{bmatrix} -I_p & 0 \\ 0 & I_q \end{bmatrix}, \qquad J_n = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix},$$

$$R_{p,q} = \frac{1}{\sqrt{2}} \begin{bmatrix} I_{p,q} & I_{p,q} \\ -I_{p,q} & I_{p,q} \end{bmatrix}, \qquad J_{p,q,r} = \begin{bmatrix} J_p & 0 \\ 0 & I_{q,r} \end{bmatrix}.$$

$J_{p,q}$ will stand for $J_{p,q,r}$ with $r = 0$.

$$K_{p,q} = \begin{bmatrix} -I_p & 0 & 0 & 0 \\ 0 & I_q & 0 & 0 \\ 0 & 0 & -I_p & 0 \\ 0 & 0 & 0 & I_q \end{bmatrix}, \qquad L_{p,q,r} = \begin{bmatrix} 0 & 0 & I_p & 0 \\ 0 & I_{q,r} & 0 & 0 \\ -I_p & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{q,r} \end{bmatrix}.$$

Also, $\mathbf{U}(n)$ is imbedded in $\mathbf{SO}(2n)$ (and $\mathbf{SO}^*(2n)$) by the mapping $A + \sqrt{-1}B \to \left[\begin{smallmatrix} A & B \\ -B & A \end{smallmatrix}\right]$ where $A + \sqrt{-1}B \in \mathbf{U}(n)$, A, B real.

We now proceed to list for u "classical", that is, $\mathfrak{su}(n)$, $\mathfrak{so}(n)$ and $\mathfrak{sp}(n)$ various σ. Tables I and II will imply that these exhaust all possibilities for σ up to conjugacy. Since the description for the fixed point Lie algebra, $\mathfrak{k}_0 = \mathfrak{l}_0 + \mathfrak{v}_0$, of σ^2 and $\mathfrak{p}_*$ is already given in [**4**, Chapter X, §2], we shall only describe $\mathfrak{l}_0$.

4.1. *The algebra* $\mathfrak{su}(n)$.

4.1.1. $\mathfrak{u} = \mathfrak{su}(2p+q)$; $\sigma(X) = J_{p,q}\bar{X}J_{p,q}^{-1}$, $p \geq 1$, $q \geq 1$. Here

$$\mathfrak{l}_0 = \left\{ \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \middle| \; A \in \mathfrak{sp}(p), \; B \in \mathfrak{so}(q) \right\}.$$

Also, since $J_{p,q}^2 = I_{2p,q}$, $\mathfrak{g}_0 = \mathfrak{su}(2p,q)$. The corresponding simply connected 4-symmetric spaces are $\mathbf{SU}(2p+q)/\mathbf{Sp}(p) \times \mathbf{SO}(q)$, $\mathbf{SU}(2p,q)/\mathbf{Sp}(p) \times \mathbf{SO}(q)$ ($p \geq 1$, $q \geq 1$). The base spaces are $\mathbf{SU}(2p+q)/\mathbf{S}(\mathbf{U}_{2p} \times \mathbf{U}_q)$, $\mathbf{SU}(2p,q)/\mathbf{S}(\mathbf{U}_{2p} \times \mathbf{U}_q)$, and the fibers are isometric to $\mathbf{S}(\mathbf{U}_{2p} \times \mathbf{U}_q)/\mathbf{Sp}(p) \times \mathbf{SO}(q)$.

4.1.2. $\mathfrak{u} = \mathfrak{su}(2n)$, $n > 2$. This Lie algebra admits an outer automorphism of order four, σ, whose fixed point Lie algebra is isomorphic to $\mathfrak{su}(n) + \mathbf{R}$. See Table II. I have not yet found an explicit description for σ. However, σ^2 must be inner, and by inspection, $\sigma^2(X) = I_{n,n}XI_{n,n}$, and σ interchanges the factors $\mathfrak{su}(n)$ of $\mathfrak{l}_0 + \mathfrak{v}_0 = \mathfrak{su}(n) + \mathfrak{su}(n) + \mathbf{R}$. Also, $\mathfrak{g}_0 = \mathfrak{su}(n,n)$.

In conclusion we have that for the simply connected 4-symmetric spaces associated with the pairs $(\mathfrak{su}(2n), \mathfrak{su}(n) + \mathbf{R})$ $(\mathfrak{su}(n,n), \mathfrak{su}(n) + \mathbf{R})$ $(n > 2)$. The base spaces are $\mathbf{SU}(2n)/\mathbf{S}(\mathbf{U}_n \times \mathbf{U}_n)$, $\mathbf{SU}(n,n)/\mathbf{S}(\mathbf{U}_n \times \mathbf{U}_n)$ and the universal cover of the fiber is isometric to $\mathbf{SU}(n)$.

4.1.3. $\mathfrak{u} = \mathfrak{su}(p+q+r+s)$; $\sigma(X) = I_{p,q,r,s} X I_{p,q,r,s}^{-1}$. Here

$$I_{p,q,r,s} = \begin{bmatrix} \sqrt{-1}I_p & 0 & 0 & 0 \\ 0 & -\sqrt{-1}I_q & 0 & 0 \\ 0 & 0 & -I_r & 0 \\ 0 & 0 & 0 & I_s \end{bmatrix}.$$

Thus

$$\mathfrak{k}_0 = \left\{ \begin{bmatrix} A & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & C & 0 \\ 0 & 0 & 0 & D \end{bmatrix} \middle| \begin{array}{l} A \in \mathfrak{u}(p),\ B \in \mathfrak{u}(q),\ C \in \mathfrak{u}(r), \\ D \in \mathfrak{u}(s),\ \mathrm{Tr}(A+B+C+D) = 0 \end{array} \right\}.$$

Also $\mathfrak{g}_0 = \mathfrak{su}\,(p+q, r+s)$. The corresponding simply connected 4-symmetric spaces are $\mathbf{SU}(p+q+r+s)/\mathbf{S}(\mathbf{U}_p \times \mathbf{U}_q \times \mathbf{U}_r \times \mathbf{U}_s)$, $\mathbf{SU}(p+q, r+s)/\mathbf{S}(\mathbf{U}_p \times \mathbf{U}_q \times \mathbf{U}_r \times \mathbf{U}_s)$ $(p \geq 1,\ r \geq 1,\ p \geq r \geq q \geq s \geq 0)$. The base spaces are

$$\mathbf{SU}(p+q+r+s)/\mathbf{S}(\mathbf{U}_{p+q} \times \mathbf{U}_{r+s}), \qquad \mathbf{SU}(p+q, r+s)/\mathbf{S}(\mathbf{U}_{p+q} \times \mathbf{U}_{r+s})$$

and the fibers are isometric to $\mathbf{S}(\mathbf{U}_{p+q} \times \mathbf{U}_{r+s})/\mathbf{S}(\mathbf{U}_p \times \mathbf{U}_q \times \mathbf{U}_r \times \mathbf{U}_s)$.

These spaces are almost Hermitian. For $q = s = 0$, they are Hermitian 2-symmetric. For $s = 0$, $q > 0$, they are almost Hermitian 3-symmetric.

4.2. *The algebra* $\mathfrak{so}\,(n)$.

4.2.1. $\mathfrak{u} = \mathfrak{so}\,(2p+q+r)$; $\sigma(X) = J_{p,q,r} X J_{p,q,r}^{-1}$. Here $\mathfrak{k}_0 = (\mathfrak{so}\,(2p) \cap \mathfrak{sp}\,(p)) \times \mathfrak{so}\,(q) \times \mathfrak{so}\,(r)$, thus $\mathfrak{k}_0$ is isomorphic to $\mathfrak{u}(p) \times \mathfrak{so}(q) \times \mathfrak{so}\,(r)$. Since $J_{p,q,r}^2 = I_{2p,q+r}$, $\mathfrak{g}_0 = \mathfrak{so}\,(2p, q+r)$.

The corresponding simply connected 4-symmetric spaces are $\mathbf{SO}(2p+q+r)/\mathbf{U}(p) \times \mathbf{SO}(q) \times \mathbf{SO}(r)$, $\mathbf{SO}_0(2p, q+r)/\mathbf{U}(p) \times \mathbf{SO}(q) \times \mathbf{SO}(r)$. The base spaces are $\mathbf{SO}(2p+q+r)/\mathbf{SO}(2p) \times \mathbf{SO}(q+r)$, $\mathbf{SO}_0(2p, q+r)/\mathbf{SO}(2p) \times \mathbf{SO}(q+r)$, and the fibers are isometric to $(\mathbf{SO}(2p)/\mathbf{U}(p)) \times (\mathbf{SO}(q+r)/\mathbf{SO}(q) \times \mathbf{SO}(r))$. If $r = 0$ and either $q = 0$ or $p = 1$, the spaces are Hermitian 2-symmetric. If $r = 0$, $p > 1$ and $q \geq 1$ the spaces are almost Hermitian 3-symmetric. If $r = 2$, $p \geq 1$ and $q \geq 1$ the spaces are almost Hermitian 4-symmetric. If $r > 2$, $p \geq 1$ and $q \geq 1$ the spaces do not admit invariant almost complex structures (cf. §6). If both r and q are odd, the automorphism is outer and again the spaces do not admit invariant almost complex structures.

Indices run as follows:

$$\begin{array}{llll} p \geq 1, & q \geq r \geq 0, & q,\ r \text{ even}, & 2p+q+r \geq 8. \\ p \geq 1, & q \geq r \geq 1, & q,\ r \text{ odd}, & 2p+q+r \geq 6. \\ p \geq 1, & q \geq 1,\ r \geq 0, & q \text{ odd },r \text{ even}, & 2p+q+r \geq 5. \end{array}$$

4.2.2. $\mathfrak{u} = \mathfrak{so}(2p+2q)$; $\sigma(X) = R_{p,q} X R_{p,q}^{-1}$. Since $R_{p,q}^2 = J_{p,q}$, it follows that $\mathfrak{k}_0$ is isomorphic to $\mathfrak{u}(p) \times \mathfrak{u}(q)$ while $\mathfrak{k}_0 + \mathfrak{v}_0 = \mathfrak{so}\,(2n) \cap \mathfrak{sp}(n)$ which is isomorphic to $\mathfrak{u}(n)$. Also $\mathfrak{g}_0 = \mathfrak{so}^*(2n)$. The corresponding simply connected 4-symmetric spaces are $\mathbf{SO}(2p+2q)/\mathbf{U}(p) \times \mathbf{U}(q)$, $\mathbf{SO}^*(2p+2q)/\mathbf{U}(p) \times \mathbf{U}(q)$ $(p \geq q \geq 1,\ p+q > 3)$. The base spaces are $\mathbf{SO}(2p+2q)/\mathbf{U}(p+q)$, $\mathbf{SO}^*(2p+2q)/\mathbf{U}(p+q)$, and the fibers are isometric to $\mathbf{U}(p+q)/\mathbf{U}(p) \times \mathbf{U}(q)$. If $q = 1$, the spaces are almost Hermitian 3-symmetric. If $q > 1$, the spaces are almost Hermitian 4-symmetric.

Note that if $q = 0$ is allowed, then the spaces are Hermitian 2-symmetric and coincide with the previous spaces in 4.2.1 for $r = q = 0$.

4.3. *The algebra* $\mathfrak{sp}(n)$.

4.3.1. $\mathfrak{u} = \mathfrak{sp}(p+q)$; $\sigma(X) = R_{p,q}XR_{p,q}^{-1}$. Here $\mathfrak{l}_0 = \mathfrak{sp}(p+q) \cap (\mathfrak{so}(2p) \times \mathfrak{so}(2q))$ which is isomorphic to $\mathfrak{u}(p) \times \mathfrak{u}(q)$. Also $\mathfrak{g}_0 = \mathfrak{sp}(p+q, \mathbf{R})$. The corresponding simply connected 4-symmetric spaces are $\mathbf{Sp}(p+q)/\mathbf{U}(p) \times \mathbf{U}(q)$, $\mathbf{Sp}(p+q, \mathbf{R})/\mathbf{U}(p) \times \mathbf{U}(q)$ $(p \geq q \geq 0,\ p+q \geq 3)$. The base spaces are $\mathbf{Sp}(p+q)/\mathbf{U}(p+q)$, $\mathbf{Sp}(p+q, \mathbf{R})/\mathbf{U}(p+q)$ and the fibers are isomorphic to $\mathbf{U}(p+q)/\mathbf{U}(p) \times \mathbf{U}(q)$. If $q = 0$, the spaces are Hermitian 2-symmetric. If $q > 0$, the spaces are almost Hermitian 4-symmetric.

4.3.2. $\mathfrak{u} = \mathfrak{sp}(p+q+r)$; $\sigma(X) = L_{p,q,r}XL_{p,q,r}^{-1}$. Here $\mathfrak{l}_0$ is isomorphic to $\mathfrak{u}(p) \times \mathfrak{sp}(q) \times \mathfrak{sp}(r)$ and since $L_{p,q,r}^2 = K_{p,q+r}$, $\mathfrak{g}_0 = \mathfrak{sp}(p, q+r)$. The corresponding simply connected 4-symmetric spaces are $\mathbf{Sp}(p+q+r)/\mathbf{U}(p) \times \mathbf{Sp}(q) \times \mathbf{Sp}(r)$, $\mathbf{Sp}(p, q+r)/\mathbf{U}(p) \times \mathbf{Sp}(q) \times \mathbf{Sp}(r)$ $(p+q+r \geq 3,\ p \geq 1,\ q \geq r \geq 0,\ q \geq 1)$. The base spaces are $\mathbf{Sp}(p+q+r)/\mathbf{Sp}(p) \times \mathbf{Sp}(q+r)$, $\mathbf{Sp}(p, q+r)/\mathbf{Sp}(p) \times \mathbf{Sp}(q+r)$. The fibers are isomorphic to $(\mathbf{Sp}(p)/\mathbf{U}(p)) \times \mathbf{Sp}(q+r)/\mathbf{Sp}(q) \times \mathbf{Sp}(r)$. If $r = 0$ and $q \geq 1$, the spaces are almost Hermitian 3-symmetric. If $q \geq r \geq 1$, the spaces do not admit invariant almost complex structures.

Note also that if $r = q = 0$ is allowed, the spaces are Hermitian 2-symmetric and coincide with the previous spaces in 4.3.1 for $q = 0$.

5. Summary and fibrations of the compact simply connected 4-symmetric spaces. Let M be a compact simply connected Riemannian 4-symmetric space. Represent M as the coset space G/L as in (2.1). Let σ be the induced automorphism of order four on G such that $(G^\sigma)_0 \subset L \subset G^\sigma$, and decompose $\mathfrak{g}$, the Lie algebra of G, as in Proposition 3.1:

$$\mathfrak{g} = \mathfrak{g}_1 + \cdots + \mathfrak{g}_r, \quad \mathfrak{l} = \mathfrak{l}_1 + \cdots + \mathfrak{l}_r, \quad \sigma = \sigma_1 + \cdots + \sigma_r,$$

where $\mathfrak{l}_i = \mathfrak{g}_i \cap \mathfrak{l}$ and σ_i (the restriction of σ to $\mathfrak{g}_i$) is an automorphism of order two or four of $\mathfrak{g}_i$ which does not preserve any proper ideals. Let $\tilde{G}_i$ be the (compact) simply connected Lie group with Lie algebra $\mathfrak{g}_i$, and let $\tilde{L}_i$ be the analytic subgroup of $\tilde{G}_i$ for $\mathfrak{l}_i$. $\tilde{L}_i$ is closed and since M is simply connected

$$M = M_1 \times \cdots \times M_r, \qquad M_i = \tilde{G}_i/\tilde{L}_i, \quad i = 1, \ldots, r. \tag{5.1}$$

Thus the classification of $M = G/L$ is reduced to the "irreducible" case considered in the next theorem. Furthermore, a description of the base space and the fiber (cf. end of §2) is also given for each "irreducible" space.

THEOREM 5.2. *Let $M = G/L$ be a compact simply connected Riemannian 4-symmetric space. Suppose that $\mathfrak{l} = \mathfrak{g}^\sigma$ where σ is an automorphism of order 4 on g which does not preserve any proper ideals. Then Tables* III, IV *and* V *give a complete list of all the possibilities, up to automorphism of G.*

REMARKS. In Table V the spaces with fiber $*$ are Hermitian 2-symmetric and hence the base space coincides with the total space and the fiber reduces to a point. The spaces marked $**$ are almost Hermitian 3-symmetric (cf. Proposition 6.3). In particular, this is one geometric feature that can be used to distinguish between the two 4-symmetric spaces associated with $\mathfrak{g}_2$. To identify the Hermitian 2-symmetric spaces and the almost Hermitian 3-symmetric spaces for G "classical", see §4.

Also, the notation here is the same as in [**4**, Chapter X, pp. 517–518].

PROOF. If the Euler characteristic of $M = G/L$ is equal to zero, then $\operatorname{rank} G > \operatorname{rank} L$ and σ is an outer automorphism of order four. Hence the pair $(\mathfrak{g}, \sigma)$ is either of Type I or of Type II or of Type III with σ outer. The first two entries in Table III account for Types I and II, while the rest of the table takes care of the outer automorphisms of order four of compact simple Lie algebras. The classification of these automorphisms is contained in Theorem 3.3.

The description of the base space and the fibers is clear for the first two cases. For G "classical", this was done in §4. For the exceptional Lie algebra $\mathfrak{e}_6$, observe that σ^2 is necessarily an inner automorphism and hence its fixed point Lie algebra is either of type $\mathfrak{su}(2) + \mathfrak{su}(6)$ or of type $\mathfrak{so}(10) + \mathbf{R}$. Then, by inspection, the entries in Table III can be verified.

Tables IV and V are related with inner automorphisms of order four of $\mathfrak{g}$. Thus we can assume that $\sigma = \mathrm{Ad}(\exp 2\pi\sqrt{-1}X)$ with X as in Proposition 3.8. Then [**20**, Proposition 2.11] $\mathfrak{g}^\sigma$ is the centralizer of a torus if and only if one of the following two conditions is satisfied:

(5.3) (a) $\mu(X) < 1$; or
(b) $\mu(X) = 1$, $n_i > 0$ implies that $m_i > 1$, and $\{m_i \colon n_i > 0\}$ is a set of $r \geq 2$ relatively prime integers.

Now if X is of the form (i), (ii) or (iii) in Proposition 3.8, then $\mu(X) < 1$ and $\mathfrak{g}^\sigma$ is the centralizer of a torus. On the other hand, if X is of the form (iv), then $X = \frac{1}{2}(V_i + V_j)$ with $m_i = m_j = 2$ or $X = V_i$ with $m_i = 4$. In either case, $\mu(X) = 1$ and condition (b) is not satisfied. Hence $\mathfrak{g}^\sigma$ is not the centralizer of a torus.

It is now easy to verify that in Table I, $\mathfrak{g}^\sigma$ is not the centralizer of a torus whenever (in the notation of the table) $-\mu$ is part of its simple root system Δ_X.

Finally we have to verify the description for the base space and the fiber.

Since we are dealing with the simply connected case, it follows that the base space is also simply connected, and that the fiber is compact and connected. Furthermore, the fibers in Table V are also simply connected, and in fact, they are Hermitian 2-symmetric (cf. §6). This explains the fact that in Table V the fibers are completely described while in Table IV only their universal cover is obtained.

If G is classical, the fibrations have been obtained in §4. For the exceptional Lie algebras the base space and the fibers are obtained by inspection (a task that is facilitated by the fact that the automorphism σ is inner). The cases where $\mathfrak{g}$ is of type $\mathfrak{g}_2$, $\mathfrak{f}_4$ or $\mathfrak{e}_8$ offer no difficulty.

If $\mathfrak{g}$ is of type $\mathfrak{e}_6$, then the only case where some ambiguity arises corresponds to the automorphism $\sigma = \mathrm{Ad}(\exp 2\pi\sqrt{-1}X)$ where $X = \frac{1}{4}(V_1 + 2V_3)$; here $\mathfrak{g}^\sigma = \mathfrak{d}_4 + \mathfrak{t}^2$, see Table I. $\mathfrak{e}_6$ has two nonconjugate inner involutions whose fixed point Lie algebras are either of type $\mathfrak{a}_1 + \mathfrak{a}_5$ or of type $\mathfrak{d}_5 + \mathfrak{t}^1$. To verify to which of these types $\mathfrak{g}^{\sigma^2}$ belongs proceed as follows:

Let α be a positive root, $\alpha = \sum c_i \alpha_i$, and let $X_\alpha \in \mathfrak{g}^\alpha$, $\mathfrak{g}^\alpha$ the root space for α. Then

$$\sigma(X_\alpha) = \exp(2\pi\sqrt{-1}\alpha(V_1 + 2V_3))X_\alpha = \exp(2\pi\sqrt{-1}\tfrac{1}{4}(c_1 + c_3))X_\alpha.$$

Thus $\sigma(X_\alpha) = -X_\alpha$ if and only if $c_1 + c_3 = 2$. The number of positive roots for which $c_1 + c_3 = 2$ is 10. Hence $\dim \mathfrak{v} = 20$ and since $\dim \mathfrak{g}^\sigma = 26$, the dimension of $\mathfrak{g}^{\sigma^2}$ is 46. This implies that $\mathfrak{g}^{\sigma^2}$ is isomorphic to $\mathfrak{d}_5 + \mathfrak{t}^1$.

If $\mathfrak{g}$ is of type $\mathfrak{e}_7$, the cases where $\mathfrak{g}^\sigma$ is of type $\mathfrak{d}_5 + \mathfrak{t}^2$ or $\mathfrak{a}_1 + \mathfrak{a}_5 + \mathfrak{t}^1$ present some ambiguity. However, a counting argument as above yields the desired results.

TABLE III. Compact Simply Connected Irreducible 4-Symmetric Spaces G/L, $\operatorname{rank} G > \operatorname{rank} L$

Total Space	Base Space	Fiber
$\{U \times U \times U \times U\}/U$ where U is compact simple and simply connected, and U is imbedded diagonally in $U \times U \times U \times U$	$\{U \times U \times U \times U\}/\{U \times U\}$ here $\{U \times U\}$ is also imbedded diagonally in $U \times U \times U \times U$	$\{U \times U\}/U$ U imbedded diagonally in $U \times U$
$\{U \times U\}/U^\theta$ where U is compact simple and simply connected, and U^θ, the fixed point set of an involution θ of U, is imbedded diagonally in $U \times U$	$(U/U^\theta) \times (U/U^\theta)$	$\{U^\theta \times U^\theta\}/U^\theta$ where U^θ is imbedded diagonally in $U^\theta \times U^\theta$
$\mathbf{SU}(2p+q)/\mathbf{Sp}(p) \times \mathbf{SO}(q)$	$\mathbf{SU}(2p+q)/\mathbf{S}(\mathbf{U}_{2p} \times \mathbf{U}_q)$	$\mathbf{S}(\mathbf{U}_{2p} \times \mathbf{U}_q)/\mathbf{Sp}(p) \times \mathbf{SO}(q)$
$(\mathfrak{su}(2n), \mathfrak{su}(n) + \mathbf{R})$	$\mathbf{SU}(2n)/\mathbf{S}(\mathbf{U}_n \times \mathbf{U}_n)$	$(\mathfrak{su}(n) + \mathfrak{su}(n), \mathfrak{su}(n))$
$\mathbf{SO}(2n)/\mathbf{U}(p) \times \mathbf{SO}(q) \times \mathbf{SO}(r)$ $2p+q+r = 2n$, both q and r odd, $n \geq 3$, $q \geq r \geq 1$	$\mathbf{SO}(2n)/\mathbf{SO}(2p) \times \mathbf{SO}(q+r)$	$(\mathbf{SO}(2p)/\mathbf{U}(p)) \times$ $\mathbf{SO}(q+r)/\mathbf{SO}(q) \times \mathbf{SO}(r)$
$(\mathfrak{e}_{6(-78)}, \mathfrak{su}(2) + \mathfrak{so}(6))$	$(\mathfrak{e}_{6(-78)}, \mathfrak{su}(2) + \mathfrak{su}(6))$	$(\mathfrak{su}(6), \mathfrak{so}(6))$
$(\mathfrak{e}_{6(-78)}, \mathfrak{so}(7) + \mathfrak{so}(3))$	$(\mathfrak{e}_{6(-78)}, \mathfrak{so}(10) + \mathbf{R})$	$(\mathfrak{so}(10), \mathfrak{so}(3) + \mathfrak{so}(7)) \times \mathbf{R}$
$(\mathfrak{e}_{6(-78)}, \mathfrak{sp}(3) + \mathbf{R})$	$(\mathfrak{e}_{6(-78)}, \mathfrak{su}(2) + \mathfrak{su}(6))$	$(\mathfrak{su}(6), \mathfrak{sp}(3)) \times (\mathfrak{su}(2), \mathbf{R})$

TABLE IV. Compact Simply Connected Irreducible 4-Symmetric Spaces G/L, $\operatorname{rank} G = \operatorname{rank} L$, L not the Centralizer of a Torus

Total Space	Base Space	Fiber Space
$\mathbf{SO}(n)/\mathbf{U}(p) \times \mathbf{SO}(q) \times \mathbf{SO}(r)$ $n = 2p+q+r$, $p \geq 1$, $q \geq 1$, $q \neq 2$ $r > 2$, r even	$\mathbf{SO}(n)/\mathbf{SO}(2p) \times \mathbf{SO}(q+r)$	$(\mathbf{SO}(2p)/\mathbf{U}(p)) \times$ $\mathbf{SO}(q+r)/\mathbf{SO}(q) \times \mathbf{SO}(r)$
$\mathbf{Sp}(n)/\mathbf{U}(p) \times \mathbf{Sp}(p) \times \mathbf{Sp}(r)$ $p+q+r = n \geq 3$, $p \geq 1$, $q \geq r \geq 1$	$\mathbf{Sp}(n)/\mathbf{Sp}(p) \times \mathbf{Sp}(q+r)$	$(\mathbf{Sp}(p)/\mathbf{U}(p)) \times$ $\mathbf{Sp}(q+r)/\mathbf{Sp}(q) \times \mathbf{Sp}(r)$
$(\mathfrak{e}_{6(-78)}, \mathfrak{so}(4) + \mathfrak{so}(6) + \mathbf{R})$	$(\mathfrak{e}_{6(-78)}, \mathfrak{so}(10) + \mathbf{R})$	$(\mathfrak{so}(10), \mathfrak{so}(4) + \mathfrak{so}(6))$
$(\mathfrak{e}_{7(-133)}, \mathfrak{so}(4) + \mathfrak{so}(8) + \mathbf{R})$	$(\mathfrak{e}_{7(-133)}, \mathfrak{so}(12) + \mathfrak{su}(2))$	$(\mathfrak{so}(12), \mathfrak{so}(4) + \mathfrak{so}(8)) \times$ $(\mathfrak{su}(2), \mathbf{R})$
$(\mathfrak{e}_{7(-133)}, \mathfrak{su}(2) + \mathfrak{su}(6) + \mathbf{R})$	$(\mathfrak{e}_{7(-133)}, \mathfrak{e}_6 + \mathbf{R})$	$(\mathfrak{e}_6, \mathfrak{su}(2) + \mathfrak{su}(6))$
$(\mathfrak{e}_{7(-133)}, \mathfrak{so}(6) + \mathfrak{so}(6) + \mathfrak{su}(2))$	$(\mathfrak{e}_{7(-133)}, \mathfrak{so}(12) + \mathfrak{su}(2))$	$(\mathfrak{so}(12), \mathfrak{so}(6) + \mathfrak{so}(6))$
$(\mathfrak{e}_{8(-248)}, \mathfrak{so}(12) + \mathfrak{su}(2) + \mathbf{R})$	$(\mathfrak{e}_{8(-248)}, \mathfrak{e}_7 + \mathfrak{su}(2))$	$(\mathfrak{e}_7, \mathfrak{so}(12) + \mathfrak{su}(2)) \times$ $(\mathfrak{su}(2), \mathbf{R})$
$(\mathfrak{e}_{8(-248)}, \mathfrak{su}(2) + \mathfrak{su}(8))$	$(\mathfrak{e}_{8(-248)}, \mathfrak{e}_7 + \mathfrak{su}(2))$	$(\mathfrak{e}_7, \mathfrak{su}(8))$
$(\mathfrak{e}_{8(-248)}, \mathfrak{so}(10) + \mathfrak{so}(6))$	$(\mathfrak{e}_{8(-248)}, \mathfrak{so}(16))$	$(\mathfrak{so}(16), \mathfrak{so}(10) + \mathfrak{so}(6))$
$(\mathfrak{f}_{4(-52)}, \mathfrak{sp}(1) + \mathfrak{sp}(2) + \mathbf{R})$	$(\mathfrak{f}_{4(-52)}, \mathfrak{sp}(3) + \mathfrak{su}(2))$	$(\mathfrak{sp}(3), \mathfrak{sp}(1) + \mathfrak{sp}(2)) \times$ $(\mathfrak{su}(2), \mathbf{R})$
$(\mathfrak{f}_{4(-52)}, \mathfrak{so}(6) + \mathfrak{so}(3))$	$(\mathfrak{f}_{4(-52)}, \mathfrak{so}(9))$	$(\mathfrak{so}(9), \mathfrak{so}(3) + \mathfrak{so}(6))$

TABLE V. Compact Simply Connected Irreducible 4-Symmetric Spaces
G/L, L the Centralizer of a Torus in G

Total Space	Base Space	Fiber Space
$\mathbf{SU}(n)/\mathbf{S}(\mathbf{U}_p \times \mathbf{U}_q \times \mathbf{U}_r \times \mathbf{U}_s)$ $n = p+q+r+s$	$\mathbf{SU}(n)/\mathbf{S}(\mathbf{U}_{p+q} \times \mathbf{U}_{r+s})$	$\mathbf{S}(\mathbf{U}_{p+q} \times \mathbf{U}_{r+s})/$ $\mathbf{S}(\mathbf{U}_p \times \mathbf{U}_q \times \mathbf{U}_r \times \mathbf{U}_s)$
$\mathbf{SO}(n)/\mathbf{U}(p) \times \mathbf{SO}(q) \times \mathbf{SO}(r)$ $n = 2p+q+r; r = 0, 2; p \geq 1$	$\mathbf{SO}(n)/\mathbf{SO}(2p) \times \mathbf{SO}(q+r)$	$(\mathbf{SO}(2p)/\mathbf{U}(p)) \times$ $\mathbf{SO}(q+r)/\mathbf{SO}(q) \times \mathbf{SO}(r)$
$\mathbf{SO}(2p+2q)/\mathbf{U}(p) \times \mathbf{U}(q)$	$\mathbf{SO}(2p+2q)/\mathbf{U}(p+q)$	$\mathbf{U}(p+q)/\mathbf{U}(p) \times \mathbf{U}(q)$
$\mathbf{Sp}(p+q)/\mathbf{U}(p) \times \mathbf{U}(q)$	$\mathbf{Sp}(p+q)/\mathbf{U}(p+q)$	$\mathbf{U}(p+q)/\mathbf{U}(p) \times \mathbf{U}(q)$
$\mathbf{Sp}(p+q)/\mathbf{U}(p) \times \mathbf{Sp}(q)$	$\mathbf{Sp}(p+q)/\mathbf{Sp}(p) \times \mathbf{Sp}(q)$	$\mathbf{Sp}(p)/\mathbf{U}(p)$
$(\mathfrak{e}_{6(-78)}, \mathfrak{so}(10) + \mathbf{R})$	$(\mathfrak{e}_{6(-78)}, \mathfrak{so}(10) + \mathbf{R})$	$\{*\}$
$(\mathfrak{e}_{6(-78)}, \mathfrak{so}(8) + \mathbf{R} + \mathbf{R})^{**}$	$(\mathfrak{e}_{6(-78)}, \mathfrak{so}(10) + \mathbf{R})$	$\mathbf{SO}(10)/\mathbf{SO}(8) \times \mathbf{SO}(2)$
$(\mathfrak{e}_{6(-78)}, \mathfrak{su}(6) + \mathbf{R})^{**}$	$(\mathfrak{e}_{6(-78)}, \mathfrak{su}(6) + \mathfrak{su}(2))$	$\mathbf{SU}(2)/\mathbf{SO}(2)$
$(\mathfrak{e}_{6(-78)}, \mathfrak{su}(5) + \mathfrak{su}(2) + \mathbf{R})^{**}$	$(\mathfrak{e}_{6(-78)}, \mathfrak{su}(6) + \mathfrak{su}(2))$	$\mathbf{SU}(6)/\mathbf{S}(\mathbf{U}_5 \times \mathbf{U}_1)$
$(\mathfrak{e}_{6(-78)}, \mathfrak{su}(5) + \mathbf{R} + \mathbf{R})$	$(\mathfrak{e}_{6(-78)}, \mathfrak{so}(10) + \mathbf{R})$	$\mathbf{SO}(10)/\mathbf{U}(5)$
$(\mathfrak{e}_{6(-78)}, \mathfrak{su}(4) + \mathfrak{su}(2) + \mathbf{R} + \mathbf{R})$	$(\mathfrak{e}_{6(-78)}, \mathfrak{su}(6) + \mathfrak{su}(2))$	$(\mathbf{SU}(2)/\mathbf{SO}(2)) \times$ $\mathbf{SU}(6)/\mathbf{S}(\mathbf{U}_4 \times \mathbf{U}_2)$
$(\mathfrak{e}_{6(-78)}, \mathfrak{su}(3) + \mathfrak{su}(3) + \mathfrak{su}(2) + \mathbf{R})$	$(\mathfrak{e}_{6(-78)}, \mathfrak{su}(6) + \mathfrak{su}(2))$	$\mathbf{SU}(6)/\mathbf{S}(\mathbf{U}_3 \times \mathbf{U}_3)$
$(\mathfrak{e}_{7(-133)}, \mathfrak{e}_6 + \mathbf{R})$	$(\mathfrak{e}_{7(-133)}, \mathfrak{e}_6 + \mathbf{R})$	$\{*\}$
$(\mathfrak{e}_{7(-133)}, \mathfrak{so}(12) + \mathbf{R})^{**}$	$(\mathfrak{e}_{7(-133)}, \mathfrak{so}(12) + \mathfrak{su}(2))$	$\mathbf{SU}(2)/\mathbf{SO}(2)$
$(\mathfrak{e}_{7(-133)}, \mathfrak{so}(10) + \mathfrak{su}(2) + \mathbf{R})^{**}$	$(\mathfrak{e}_{7(-133)}, \mathfrak{so}(12) + \mathfrak{su}(2))$	$\mathbf{SO}(12)/\mathbf{SO}(10) \times \mathbf{SO}(2)$
$(\mathfrak{e}_{7(-133)}, \mathfrak{su}(7) + \mathbf{R})^{**}$	$(\mathfrak{e}_{7(-133)}, \mathfrak{su}(8))$	$\mathbf{SU}(8)/\mathbf{S}(\mathbf{U}_7 \times \mathbf{U}_1)$
$(\mathfrak{e}_{7(-133)}, \mathfrak{so}(10) + \mathbf{R} + \mathbf{R})$	$(\mathfrak{e}_{7(-133)}, \mathfrak{e}_6 + \mathbf{R})$	$(\mathfrak{e}_{6(-78)}, \mathfrak{so}(10) + \mathbf{R})$
$(\mathfrak{e}_{7(-133)}, \mathfrak{su}(6) + \mathbf{R} + \mathbf{R})$	$(\mathfrak{e}_{7(-133)}, \mathfrak{so}(2) + \mathfrak{so}(12))$	$(\mathbf{SO}(12)/\mathbf{U}(6)) \times$ $\mathbf{SU}(2)/\mathbf{SO}(2)$
$(\mathfrak{e}_{7(-133)}, \mathfrak{su}(2) + \mathfrak{su}(6) + \mathbf{R})$	$(\mathfrak{e}_{7(-133)}, \mathfrak{su}(2) + \mathfrak{so}(12))$	$\mathbf{SO}(12)/\mathbf{U}(6)$
$(\mathfrak{e}_{7(-133)}, \mathfrak{su}(5) + \mathfrak{su}(3) + \mathbf{R})$	$(\mathfrak{e}_{7(-133)}, \mathfrak{su}(8))$	$\mathbf{SU}(8)/\mathbf{S}(\mathbf{U}_5 \times \mathbf{U}_3)$
$(\mathfrak{e}_{8(-248)}, \mathfrak{so}(14) + \mathbf{R})^{**}$	$(\mathfrak{e}_{8(-248)}, \mathfrak{so}(16))$	$\mathbf{SO}(16)/\mathbf{SO}(14) \times \mathbf{SO}(2)$
$(\mathfrak{e}_{8(-248)}, \mathfrak{e}_7 + \mathbf{R})^{**}$	$(\mathfrak{e}_{8(-248)}, \mathfrak{su}(2) + \mathfrak{e}_7)$	$\mathbf{SU}(2)/\mathbf{SO}(2)$
$(\mathfrak{e}_{8(-248)}, \mathfrak{su}(8) + \mathbf{R})$	$(\mathfrak{e}_{8(-248)}, \mathfrak{so}(16))$	$\mathbf{SO}(16)/\mathbf{U}(8)$
$(\mathfrak{e}_{8(-248)}, \mathfrak{su}(2) + \mathfrak{e}_6 + \mathbf{R})$	$(\mathfrak{e}_{8(-248)}, \mathfrak{su}(2) + \mathfrak{e}_7)$	$(\mathfrak{e}_{7(-133)}, \mathfrak{e}_6 + \mathbf{R})$
$(\mathfrak{f}_{4(-52)}, \mathfrak{sp}(3) + \mathbf{R})^{**}$	$(\mathfrak{f}_{4(-52)}, \mathfrak{sp}(3) + \mathfrak{su}(2))$	$\mathbf{SU}(2)/\mathbf{SO}(2)$
$(\mathfrak{f}_{4(-52)}, \mathfrak{so}(7) + \mathbf{R})^{**}$	$(\mathfrak{f}_{4(-52)}, \mathfrak{so}(9))$	$\mathbf{SO}(9)/\mathbf{SO}(7) \times \mathbf{SO}(2)$
$(\mathfrak{f}_{4(-52)}, \mathfrak{su}(2) + \mathfrak{su}(3) + \mathbf{R})$	$(\mathfrak{f}_{4(-52)}, \mathfrak{su}(2) + \mathfrak{sp}(3))$	$\mathbf{Sp}(3)/\mathbf{U}(3)$
$(\mathfrak{g}_{2(-14)}, \mathfrak{su}(2) + \mathbf{R})^{**}$	$(\mathfrak{g}_{2(-14)}, \mathfrak{su}(2) + \mathfrak{su}(2))$	$\mathbf{SU}(2)/\mathbf{SO}(2)$
$(\mathfrak{g}_{2(-14)}, \mathfrak{su}(2) + \mathbf{R})$	$(\mathfrak{g}_{2(-14)}, \mathfrak{su}(2) + \mathfrak{su}(2))$	$\mathbf{SU}(2)/\mathbf{SO}(2)$

To conclude this section, note that in the decomposition (5.1), the possibility that M_i be 2-symmetric is not ruled out. Thus Theorem 5.2 can be completed as follows:

THEOREM 5.4. *The compact simply connected "irreducible" Riemannian 4-symmetric spaces are the irreducible Riemannian 2-symmetric spaces and the spaces listed in Tables* III, IV *and* V.

6. Almost Hermitian 4-symmetric spaces.

DEFINITION. Let M be a Riemannian n-symmetric space; M is said to be an almost Hermitian n-symmetric space if M admits an almost Hermitian structure invariant under the symmetries of M.

We already have a large class of examples of these spaces. On one hand, every Hermitian 2-symmetric space is also Hermitian n-symmetric for any $n > 1$. On the other hand, it is well known that every homogeneous space G/L with G compact and L the centralizer of a torus admits G-invariant almost complex structures, in fact, it admits a G-invariant almost complex structure that makes the space G/L into a Hodge manifold (see, e.g. [**20**, Corollary 9.5]). Thus, in particular, the spaces in Table V are (almost) Hermitian 4-symmetric. The central result of this section is that these spaces are essentially all the irreducible compact almost Hermitian 4-symmetric spaces. More precisely, we prove the following:

THEOREM 6.1. *Let M be a compact connected almost Hermitian 4-symmetric space. Then M globally is the direct product of a torus and spaces listed in Table* V, *where each factor is a totally geodesic almost Hermitian 4-symmetric submanifold.*

PROOF. Assume first that M has no Euclidean factor, and let $\tilde{M}$ be the universal covering space of M. $\tilde{M}$ is also compact, and $\tilde{M} = M_1 \times \cdots \times M_r$ as in (5.1). By invariance, the almost complex structure preserves the tangent space of the fibers. Thus, the fiber of each M_i must be Hermitian 2-symmetric. A glance at Tables III, IV and V shows that this is possible only for the spaces in Table V. Hence, each M_i has nonvanishing Euler characteristic, and so does $\tilde{M}$. It follows that M has nonvanishing Euler characteristic and therefore that it is simply connected as well (cf. [**20**, Proposition 4.1]). Thus $M = \tilde{M} = M_1 \times \cdots \times M_r$ with each M_i listed in Table V.

Consider now the case where M has a Euclidean factor. Let $\tilde{M}$ be its universal covering space, and decompose $\tilde{M}$ as the product $\tilde{M} = M_0 \times M_1 \times \cdots \times M_r$ where M_0 is an Euclidean space, and the M_i $(1 \leq i \leq r)$ are as in (5.1), and hence listed in Table V. Then (following J. A. Wolf [**18**, Lemma 1]) $M = \tilde{M}/\Gamma$, where Γ is the group of deck transformations. Γ acts freely and properly discontinuously on $\tilde{M}$. Furthermore, Γ preserves each M_i (see [**17**, Theorem 3.1.2]). Then Γ acts on M_0 as a group of pure translations, and acts trivially on the other M_i.

REMARK. The separation of the Euclidean factor is due to the referee. He also suggested to find a proof that avoids the classification and thus obtain a general theorem for n-symmetric spaces, any $n > 1$. Unfortunately, the author has not succeeded in this point.

An immediate consequence of Theorem 6.1 is the following.

COROLLARY 6.2. *Let M be a compact connected almost Hermitian 4-symmetric space. Represent M as the coset space G/L as in (2.1). If M has positive Euler characteristic, then L is the centralizer of a torus in G.*

Riemannian n-symmetric spaces with n odd are in a natural way almost Hermitian n-symmetric [**14**, §6]. However, it is no longer true, in general, that these spaces will be m-symmetric for $m > n$. We shall now determine the class of compact 3-symmetric spaces which are also 4-symmetric.

The classification of 3-symmetric spaces has been accomplished by J. Wolf and A. Gray [**20**], see also [**4**, p. 583]. They are homogeneous spaces G/L with $(G^\theta)_0 \subset L \subset G^\theta$, θ an automorphism of order 3 of G. Roughly, these spaces can be divided into three categories: (i) $\operatorname{rank} L < \operatorname{rank} G$, (ii) $\operatorname{rank} L = \operatorname{rank} G$, L not the centralizer of a torus, and (iii) L the centralizer of a torus. Due to Corollary 6.2, the 3-symmetric spaces falling into categories (i) and (ii) cannot be 4-symmetric. On the other hand, the spaces in the third category are good candidates to be almost Hermitian 4-symmetric. The following proposition shows that this is indeed so. Since such a space M decomposes as a product of "simple" factors $M = M_1 \times \cdots \times M_r$, $M_i = G_i/L_i$ with G_i simple, we only have to consider this case.

PROPOSITION 6.3. *Let M be a 3-symmetric space G/L, where G is a compact simple Lie group acting effectively on M and L is the centralizer of a torus. Suppose $\mathfrak{l} = \mathfrak{g}^\phi$ where ϕ is an automorphism of order 3. Then M is also an almost Hermitian 4-symmetric space.*

PROOF. (The notation here is as in 3.6.) According to [**20**, Proposition 3.3] if ϕ is an inner automorphism of order 3 of $\mathfrak{g}$, then ϕ is conjugate to some $\theta = \operatorname{Ad}(\exp 2\pi\sqrt{-1}X)$ where either $X = \frac{1}{3}m_iV_i$, $1 \leq m_i \leq 3$ or $X = \frac{1}{3}(V_i + V_j)$ with $m_i = m_j = 1$. Since it is assumed that $\mathfrak{l} = \mathfrak{g}^\phi$ is the centralizer of a torus then necessarily $X = \frac{1}{3}m_iV_i$ with $m_i = 1, 2$ or $X = \frac{1}{3}(V_i + V_j)$ with $m_i = m_j = 1$. (To see this use (5.3).) In either case, if $\sigma = \operatorname{Ad}(\exp 2\pi\sqrt{-1}\frac{3}{4}X)$, then σ is an automorphism of order four whose fixed point set $\mathfrak{g}^\sigma$ coincides with $\mathfrak{g}^\theta$. Hence, the corresponding space G/L is also a 4-symmetric space.

In particular, this shows that the spaces marked ** in Table V are 3-symmetric.

A similar argument proves the following:

PROPOSITION 6.4. *Let M be a compact almost Hermitian 4-symmetric space. Then M is also an almost Hermitian n-symmetric space for any $n \geq 4$.*

One final observation that deserves special attention is the following:

PROPOSITION 6.5. *Let M be the 3-symmetric space $\mathbf{SO}(2n+2)/\mathbf{U}(n) \times \mathbf{U}(1)$. Then M admits two nonequivalent 4-symmetric structures whose fibrations are described as follows:*

Base Space	Fiber Space
$\mathbf{SO}(2n+2)/\mathbf{U}(n+1)$	$\mathbf{U}(n+1)/\mathbf{U}(n) \times \mathbf{U}(1)$
$\mathbf{SO}(2n+2)/\mathbf{SO}(2n) \times \mathbf{SO}(2)$	$\mathbf{SO}(2n)/\mathbf{U}(n)$

PROOF. See §4, 4.2.

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